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Two-loop three-point diagrams with irreducible numerators

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Abstract

We study the problem of calculating two-loop three-point diagrams with irreducible numerators (i.e. numerators which cannot be expressed in terms of the denominators). For the case of massless internal particles and arbitrary (off-shell) external momenta, exact results are obtained in terms of polylogarithms. We also consider the tensor decomposition of two-loop three-point diagrams, and show how it is connected with irreducible numerators.

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1. The study of two-loop diagrams is required by increasing precision of experiments testing QCD and the Standard Model. As compared with the one-loop case, the calculation of two-loop diagrams is technically much more complicated. Exact results are known for a few special cases only, mainly for two-point functions. There are, however, some approaches which make it possible to obtain numerical results [1, 2] or analytic expansions [3] providing reasonable precision for the cases of interest.

Three-point two-loop diagrams are much less investigated. In this case, we have two independent external momenta (from which three independent invariants can be constructed). Although some of the numerical approaches mentioned above (see also in [4]) can also be applied to three-point functions, this usually requires a larger number of parametric integrations to be done numerically, and the structure of the singularities in the integrand is more complicated. Asymptotic expansions can also be constructed, but they involve three-fold series in external momentum invariants (see, e.g., in [2]). Furthermore, the expansion at large external momenta requires more information about three-point diagrams with massless internal lines involving not only higher powers of the propagators, but also some numerators which cannot be cancelled against any of the denominators. We shall refer to these as "irreducible numerators". One also obtains them in realistic calculations with vector and spinor particles. Moreover, the problem of tensor decomposition of integrals with uncontracted Lorentz indices is also closely connected with irreducible numerators. This is one of the essential complications of two-loop three-point calculations as compared with both one-loop case [5, 6] and two-point self-energies [7].

The problem of irreducible numerators (and the related problem of tensor reduction) is the main subject to be discussed in the present paper. In fact, it can be considered as a continuation of the papers [8, 9] where some exact results for three-point two-loop diagrams were derived[3]. The remainder of the paper is organized as follows. In section 2 we collect some useful results for one-loop triangle diagrams. In section 3 we present some results for two-loop three-point functions and discuss the problem of irreducible numerators. Then, we calculate integrals with irreducible numerators for the planar (section 4) and non-planar (section 5) cases. In section 6 we consider the tensor decomposition of three-point two-loop diagrams. In section 7 we discuss the results.

2. Here we shall briefly summarize some useful formulae for one-loop integrals.

Definition. The n-dimensional one-loop three-point Feynman integral is defined as

$$J(n;\ \nu_1,\nu_2,\nu_3|\ p_1^2,p_2^2,p_3^2) \equiv \int \frac{\mathrm{d}^n q}{((p_2-q)^2)^{\nu_1}((p_1+q)^2)^{\nu_2}(q^2)^{\nu_3}}, \tag{1}$$

where all external momenta p_1,p_2,p_3 (such that $p_1+p_2+p_3=0$) are ingoing, and the "causal" prescription $(q^2)^{-\nu} \leftrightarrow (q^2+\mathrm{i}0)^{-\nu}$ is understood. Below we shall omit the arguments p_1^2,p_2^2,p_3^2 in J.

"Uniqueness" relations. At special values of the sum of the powers of denominators, $\sum\nu_i \equiv \nu_1+\nu_2+\nu_3$, the following formulae [12, 13] are valid:

$$J(n;\ \nu_1,\nu_2,\nu_3)\Big|_{\Sigma\nu_i=n} = \pi^{n/2}\ \mathrm{i}^{1-n}\prod_{i=1}^{3}\frac{\Gamma(n/2-\nu_i)}{\Gamma(\nu_i)}\ \frac{1}{(p_i^2)^{n/2-\nu_i}}, \tag{2}$$

[3]In the papers [10, 11] some results were generalized to the case of an arbitrary number of loops.

$$\left\{ \nu_1 J(n;\ \nu_1+1,\nu_2,\nu_3) + \nu_2 J(n;\ \nu_1,\nu_2+1,\nu_3) + \nu_3 J(n;\ \nu_1,\nu_2,\nu_3+1) \right\}\Big|_{\Sigma\nu_i = n-2}$$
$$= \pi^{n/2}\ \mathrm{i}^{1-n} \prod_{i=1}^{3} \frac{\Gamma(n/2-\nu_i-1)}{\Gamma(\nu_i)}\ \frac{1}{(p_i^2)^{n/2-\nu_i-1}}. \qquad (3)$$

Note that the sum of the powers of inverse momenta squared on the r.h.s. of (2) is $n/2$.

The case $\nu_1=\nu_2=\nu_3=1,\ n=4$. The result for this case is well-known (see, e.g., in [15, 16]). Following the notation of refs. [14, 8], we write it as

$$C^{(1)}(p_1^2,p_2^2,p_3^2) \equiv J(4;1,1,1) = \mathrm{i}\pi^2\ (p_3^2)^{-1}\ \Phi^{(1)}(x,y), \qquad (4)$$

$$x \equiv p_1^2/p_3^2 \quad , \quad y \equiv p_2^2/p_3^2\ . \qquad (5)$$

The function $\Phi^{(1)}$ can be represented as a parametric integral,

$$\Phi^{(1)}(x,y) = -\int\limits_0^1 \frac{\mathrm{d}\xi}{y\xi^2+(1-x-y)\xi+x}\ \left(\ln\frac{y}{x} + 2\ln\xi\right), \qquad (6)$$

or in terms of dilogarithms,

$$\Phi^{(1)}(x,y) = \frac{1}{\lambda}\left\{ 2\left(\mathrm{Li}_2\left(-\rho x\right) + \mathrm{Li}_2\left(-\rho y\right)\right) + \ln\frac{y}{x}\ln\frac{1+\rho y}{1+\rho x} + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3}\right\}, \quad (7)$$

$$\lambda(x,y) \equiv \sqrt{(1-x-y)^2-4xy} \quad , \quad \rho(x,y) \equiv 2\ (1-x-y+\lambda)^{-1}. \qquad (8)$$

The case $\nu_1=\nu_2=\nu_3=1,\ n=4-2\varepsilon$. In ref. [9], the following representation was obtained for arbitrary $n=4-2\varepsilon$:

$$J(4-2\varepsilon;1,1,1) = \frac{\pi^{2-\varepsilon}\ \mathrm{i}^{1+2\varepsilon}}{(p_3^2)^{1+\varepsilon}}\ \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\ \frac{1}{\varepsilon}\int\limits_0^1 \frac{\mathrm{d}\xi\ \xi^{-\varepsilon}\left((y\xi)^{-\varepsilon}-(x/\xi)^{-\varepsilon}\right)}{\left(y\xi^2+(1-x-y)\xi+x\right)^{1-\varepsilon}}. \quad (9)$$

The expansion in ε yields

$$J(4-2\varepsilon;1,1,1) = \pi^{2-\varepsilon}\ \mathrm{i}^{1+2\varepsilon}(p_3^2)^{-1-\varepsilon}\ \Gamma(1+\varepsilon)\ \left\{\Phi^{(1)}(x,y) + \varepsilon\ \Psi^{(1)}(x,y) + \mathcal{O}(\varepsilon^2)\right\}, \quad (10)$$

where $\Phi^{(1)}$ is defined by (6)–(7) while $\Psi^{(1)}$ can be represented as

$$\Psi^{(1)}(x,y) = -\int\limits_0^1 \frac{\mathrm{d}\xi}{y\xi^2+(1-x-y)\xi+x}\ \left\{\left(\ln\frac{y}{x}+2\ln\xi\right)\ln(y\xi^2+(1-x-y)\xi+x)\right.$$
$$\left. -2\ln y\ln\xi - 2\ln^2\xi - \frac{1}{2}\ln(xy)\ln\frac{y}{x}\right\}. \qquad (11)$$

The result in terms of polylogarithms (up to the third order) is presented in [9], eq. (29).

The case $\nu_1=\nu_2=1,\ \nu_3=1+\delta,\ n=4$. In ref. [8], the following formula was derived:

$$J(4;1,1,1+\delta) = \frac{\mathrm{i}\pi^2}{(p_3^2)^{1+\delta}}\ \frac{1}{\delta}\int\limits_0^1 \mathrm{d}\xi \frac{(y\xi)^{-\delta}-(x/\xi)^{-\delta}}{y\xi^2+(1-x-y)\xi+x}. \qquad (12)$$

The first terms of the expansion of $J(4;1,1,1+\delta)$ in δ, up to and including δ^2, are

$$\frac{\mathrm{i}\pi^2}{(p_3^2)^{1+\delta}}\left\{\left(1-\frac{1}{2}\delta(\ln x+\ln y) + \frac{1}{6}\delta^2(\ln^2 x+\ln x\ln y+\ln^2 y)\right)\Phi^{(1)}(x,y) + \frac{1}{3}\delta^2\Phi^{(2)}(x,y)\right\} (13)$$

where $\Phi^{(2)}$ is connected with the two-loop planar diagram (see in Section 3).

Note that in all integral representations (6), (9), (11) and (12) the denominator of the integrand can be represented as a propagator, $p_3^2\,(y\xi^2+(1-x-y)\xi+x)=(p_1+\xi p_2)^2$.

3 There are two basic "topologies" of the two-loop three-point diagrams: planar (Fig. 1a) and non-planar (Fig. 1b) ones. Note that the diagram in Fig. 1b is symmetric with respect to all three external lines, while the diagram in Fig. 1a is symmetric with respect to the two lower lines only. The corresponding Feynman integrals are:

$$C^{(2)}(p_1^2,p_2^2,p_3^2)=\int\int\frac{\mathrm{d}q\;\mathrm{d}r}{(p_1+r)^2\;(p_1+q)^2\;(p_2-r)^2\;(p_2-q)^2\;r^2\;(q-r)^2}, \tag{14}$$

$$\widetilde{C}^{(2)}(p_1^2,p_2^2,p_3^2)=\int\int\frac{\mathrm{d}q\;\mathrm{d}r}{(p_1+q)^2\;(p_1+q+r)^2\;(p_2-r)^2\;(p_2-q-r)^2\;r^2\;q^2} \tag{15}$$

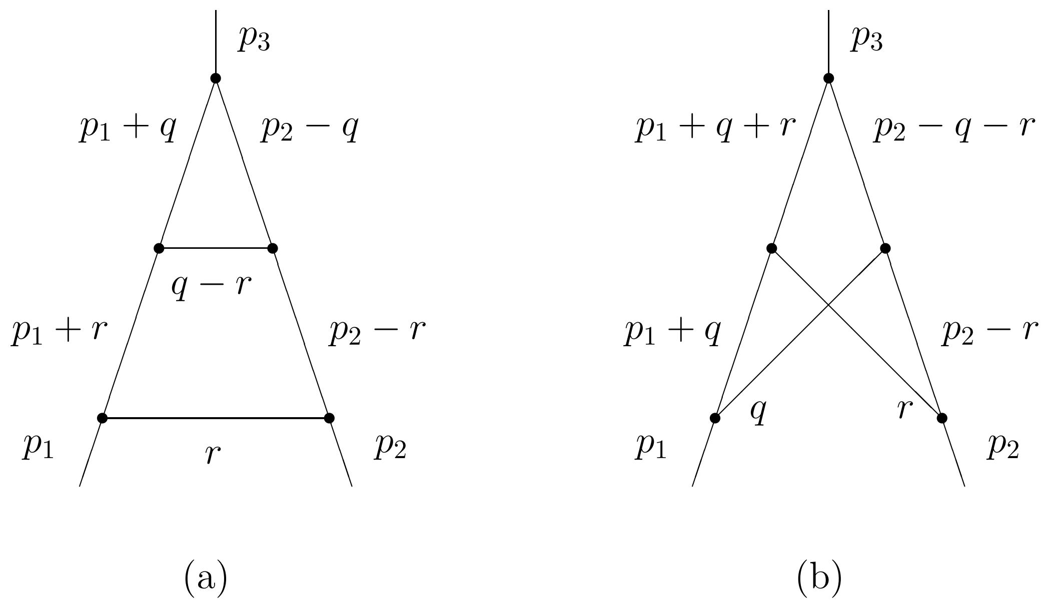

Figure 1: Two-loop three-point diagrams: planar (a) and non-planar (b) cases

The planar diagram (14) was calculated (in four dimensions) in ref. [8]. The result is

$$C^{(2)}(p_1^2,p_2^2,p_3^2)=(\mathrm{i}\pi^2)^2\;(p_3^2)^{-2}\;\Phi^{(2)}(x,y), \tag{16}$$

where the function $\Phi^{(2)}$ (see also (13)) is

$$\Phi^{(2)}(x,y)=-\frac{1}{2}\int\limits_0^1\frac{\mathrm{d}\xi}{y\xi^2+(1-x-y)\xi+x}\;\ln\xi\left(\ln\frac{y}{x}+\ln\xi\right)\left(\ln\frac{y}{x}+2\ln\xi\right). \tag{17}$$

It can be expressed in terms of polylogarithms up to the fourth order, see e.g. [10], eq. (19)[4].

[4]We would like to note that recently this result was checked numerically [17].

As to the non-planar diagram (15), in refs. [18, 9] it was obtained (also at $n=4$) that

$$\widetilde{C}^{(2)}(p_1^2,p_2^2,p_3^2) = \left(C^{(1)}(p_1^2,p_2^2,p_3^2)\right)^2 = (\mathrm{i}\pi^2)^2 \; (p_3^2)^{-2} \; \left(\Phi^{(1)}(x,y)\right)^2 , \tag{18}$$

with $\Phi^{(1)}(x,y)$ defined by (6)–(7). Therefore, $\widetilde{C}^{(2)}$ can be expressed in terms of dilogarithms and their products.

Now, let us consider the cases when some numerators occur in the integrands on r.h.s.'s of (14), (15). First of all, let us introduce the following notation:

$$C^{(2)}\,[\text{something}] \equiv \{\text{integral (14) with [something] in the numerator}\}\,, \tag{19}$$

$$\widetilde{C}^{(2)}\,[\text{something}] \equiv \{\text{integral (15) with [something] in the numerator}\}\,. \tag{20}$$

In this notation, eqs. (14) and (15) correspond to $C^{(2)}[1]$ and $\widetilde{C}^{(2)}[1]$, respectively. Inserting some numerators into (14) and (15) may produce divergent integrals. In these cases, we shall understand that dimensional regularization [19] is employed to regulate these singularities[5], and (for two-loop integrals) we shall usually omit terms vanishing as $\varepsilon \to 0$.

In the paper [9] some integrals of the type of (19), (20) were considered, and the results can be written (in the new notation) as

$$C^{(2)}\left[(p_1+r)^2\right] = \widetilde{C}^{(2)}\left[(p_1+q)^2\right] = \widetilde{C}^{(2)}\left[q^2\right] = p_1^2\; C^{(2)}(p_2^2,p_3^2,p_1^2) = \frac{(\mathrm{i}\pi^2)^2}{p_1^2}\;\Phi^{(2)}\left(\frac{1}{x},\frac{y}{x}\right), \tag{21}$$

$$\widetilde{C}^{(2)}\left[(p_1+q+r)^2\right] = \widetilde{C}^{(2)}\left[(p_2-q-r)^2\right] = p_3^2\; C^{(2)}(p_1^2,p_2^2,p_3^2) = \frac{(\mathrm{i}\pi^2)^2}{p_3^2}\;\Phi^{(2)}(x,y), \tag{22}$$

$$C^{(2)}\left[(q-r)^2\right] = \frac{\mathrm{i}^{2+4\varepsilon}\pi^{4-2\varepsilon}}{(p_3^2)^{1+2\varepsilon}}\;\frac{\Gamma^2(1+\varepsilon)}{1-2\varepsilon}\;\left\{\frac{1}{\varepsilon}\Phi^{(1)}(x,y)+\Psi^{(1)}(x,y)\right\}, \tag{23}$$

$$C^{(2)}\left[(p_1+q)^2\right] = \frac{\mathrm{i}^{2+4\varepsilon}\pi^{4-2\varepsilon}}{(p_1^2)^{1+2\varepsilon}}\;\frac{\Gamma^2(1+\varepsilon)}{1-2\varepsilon}\left\{\left(\frac{1}{\varepsilon}-\frac{1}{2}\ln\frac{y}{x^2}\right)\Phi^{(1)}\left(\frac{1}{x},\frac{y}{x}\right)+\Psi^{(1)}\left(\frac{1}{x},\frac{y}{x}\right)\right\}, \tag{24}$$

$$\begin{aligned} C^{(2)}&\left[(p_1+r)^2(p_2-r)^2\right] = \widetilde{C}^{(2)}\left[(p_1+q)^2(p_2-r)^2\right] \\ &= \frac{\mathrm{i}^{2+4\varepsilon}\pi^{4-2\varepsilon}}{(p_3^2)^{2\varepsilon}}\;\frac{\Gamma^2(1+\varepsilon)}{2(1-2\varepsilon)(1-3\varepsilon)}\;\left\{\frac{1}{\varepsilon^2}-\ln x\ln y+(1-x-y)\Phi^{(1)}(x,y)-\frac{\pi^2}{3}\right\}. \end{aligned} \tag{25}$$

One should remember that "$+\mathcal{O}(\varepsilon)$" is understood in all r.h.s.'s. Some other results can be obtained from (21)–(25) by using symmetry properties, or they may correspond to propagator-type diagrams, for example (see [20])

$$C^{(2)}\left[r^2\right] = (\mathrm{i}\pi^2)^2\;(p_3^2)^{-1}\;6\zeta(3). \tag{26}$$

In all these examples (21)–(26) the numerators can be cancelled against the corresponding denominators, and (effectively) we obtain diagrams where some of the lines

[5] For simplicity, we put the dimensional regularization scale parameter $\mu_0 = 1$.

are reduced to points. There exist, however, some scalar numerators which cannot be represented in terms of the denominators of the diagrams in Fig. 1a,b; namely:

$$C^{(2)}\left[q^2\right] \qquad \text{and} \qquad \widetilde{C}^{(2)}\left[(q+r)^2\right]. \tag{27}$$

Other irreducible cases can be related to these two ones. It is easy to see that, if we introduce auxiliary "forward scattering" four-point functions according to Fig. 2a and Fig. 2b, these numerators will correspond to the dashed lines (missing in Fig. 1a and Fig. 1b). Calculation of these integrals (27) will be considered in the next two sections.

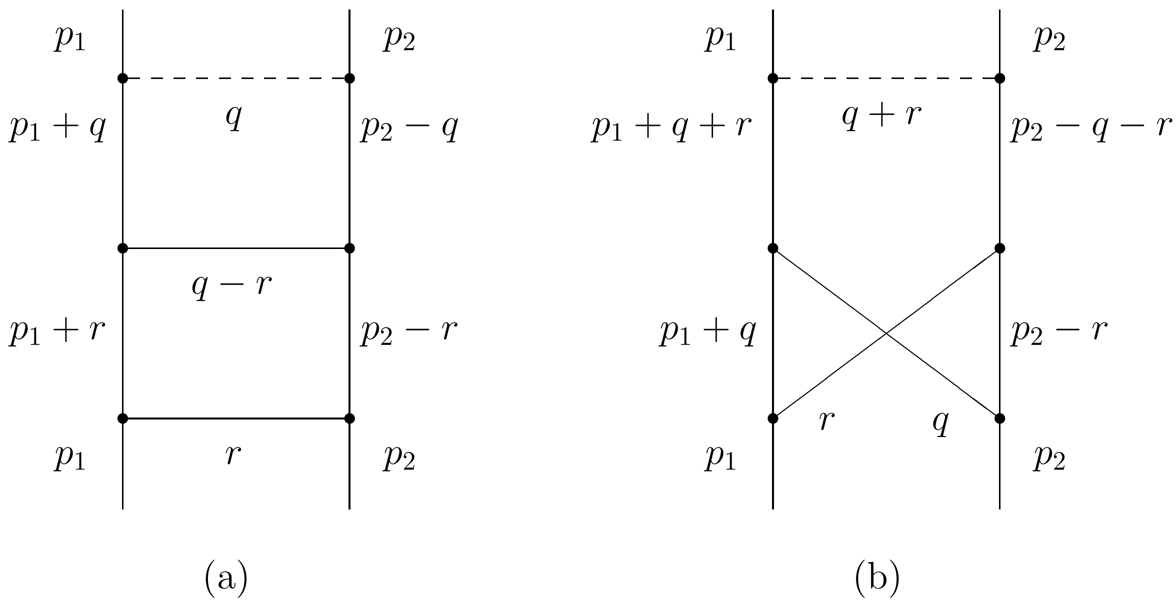

Figure 2: Auxiliary planar (a) and non-planar (b) four-point functions

4. In this section we shall examine the planar diagram with irreducible numerator, $C^{(2)}\left[q^2\right]$. It is easy to see that it can be written as

$$C^{(2)}\left[q^2\right] = C^{(2)}\left[(p_1+q)^2\right] - C^{(2)}\left[(p_1+r)^2\right] + C^{(2)}\left[r^2\right] - 2p_1^\mu\, C^{(2)}\left[(q-r)_\mu\right], \tag{28}$$

where only the first term on the r.h.s. is singular in ε. All the terms on the r.h.s. are known (see eqs. (25), (21), (26)), exept the last one, $p_1^\mu\, C^{(2)}\left[(q-r)_\mu\right]$, which is convergent.

Let us consider $p_1^\mu\, C^{(2)}\left[(q-r)_\mu\right]$ at $n=4$ and introduce auxiliary analytic regularization by multiplying the integrand with $[(p_1+r)^2]^{-\delta}[(p_2-r)^2]^{-\delta}[r^2]^{\delta}[(q-r)^2]^{\delta}$. Since the integral is convergent, the result should correspond to the limit $\delta\to 0$. Then, let us consider q-integration and integrate by parts[6] (see in [21]):

$$p_1^\mu \int \frac{\mathrm{d}q\ (q-r)_\mu}{(p_1+q)^2\ (p_2-q)^2\ [(q-r)^2]^{1-\delta}} = \frac{1}{2\delta}\, p_1^\mu \int \frac{\mathrm{d}q}{(p_1+q)^2\ (p_2-q)^2}\ \frac{\partial}{\partial q^\mu}[(q-r)^2]^\delta$$

[6]The calculation of $C^{(2)}\left[q^2\right]$ can alternatively be considered in the coordinate space.

$$= \frac{1}{\delta} \int \frac{\mathrm{d}q \; (p_1, p_1+q)}{[(p_1+q)^2]^2 \; (p_2-q)^2 \; [(q-r)^2]^{-\delta}} - \frac{1}{\delta} \int \frac{\mathrm{d}q \; (p_1, p_2-q)}{(p_1+q)^2 \; [(p_2-q)^2]^2 \; [(q-r)^2]^{-\delta}}. \quad (29)$$

Now, let us study the contributions corresponding to the two integrals on the r.h.s. separately. Each of them is sungular in δ, but together they should give a finite result.

In the first contribution, the sum of the powers of $(p_2-r)^2, (p_2-q)^2$ and $(q-r)^2$ in the denominator is equal to two (half the space-time dimension if $n=4$). So, one can try to use the "uniqueness" relation (2) to transform the product of these propagators into a triangle. However, we need to introduce an additional regularization by multiplying the integrand with $[(p_1+r)^2]^{-\delta'}[(p_2-q)^2]^{-\delta'}[(q-r)^2]^{\delta'}$ (otherwise we would obtain "bare" singularities in separate terms). Since the contribution we are considering is regular in δ' (but not in δ), we need to remember that, after the calculation is performed, we have to let $\delta' \to 0$ first. Now, we perform several transformations by using the relations (2) and (3) (and also trivial formulae for one-loop two-point functions), and we arrive at the following representation of this contribution:

$$\frac{1}{\delta} \int\int \frac{\mathrm{d}q \; \mathrm{d}r \; (p_1, p_1+q)}{[(p_1+r)^2]^{1+\delta+\delta'} \; [(p_1+q)^2]^2 \; [(p_2-r)^2]^{1+\delta} \; [(p_2-q)^2]^{1+\delta'} \; [r^2]^{1-\delta} \; [(q-r)^2]^{-\delta-\delta'}}$$
$$= \frac{\mathrm{i}\pi^2}{2\delta^2(1+\delta)} \left\{ -\mathrm{i}\pi^2 \; (p_3^2)^{-1-\delta'} \; \frac{1}{\delta'} - (1+\delta')p_1^2 \; J(4;1,2+\delta',1) \right.$$
$$\left. + \frac{1}{\delta'} (p_1^2)^{1+\delta} \left[(\delta+\delta')(1+\delta+\delta') \; J(4;1,2+\delta+\delta',1) - \delta(1+\delta) \; J(4;1,2+\delta,1) \right] \right\}, \quad (30)$$

where the integrals J are one-loop triangles defined by (1).

Considering the second term on the r.h.s. of eq. (29) and using the fact that, due to momentum conservation, $-(p_1, p_2-q) = (p_2, p_2-q) + (p_3, p_2-q)$, it is easy to see that the term (p_2, p_2-q) should give the same as (30) with $p_1 \leftrightarrow p_2$. To calculate the term corresponding to (p_3, p_2-q), we need to use only the relation (2) (and we do not need to introduce additional parameter δ'). In such a way, we get

$$\frac{1}{\delta} \int\int \frac{\mathrm{d}q \; \mathrm{d}r \; (p_3, p_2-q)}{[(p_1+r)^2]^{1+\delta} \; (p_1+q)^2 \; [(p_2-r)^2]^{1+\delta} \; [(p_2-q)^2]^2 \; [r^2]^{1-\delta} \; [(q-r)^2]^{-\delta}}$$
$$= -\frac{\mathrm{i}\pi^2}{2\delta(1+\delta)} \left\{ J(4;1,1,1) + (p_2^2)^{\delta} \; J(4;1+\delta,1,1) \right\}$$
$$-\frac{1}{2} \int\int \frac{\mathrm{d}q \; \mathrm{d}r}{[(p_1+q)^2]^{1+\delta} \; [(p_2-r)^2]^{1+\delta} \; [(p_2-q)^2]^{1-\delta} \; [r^2]^{1-\delta} \; (q-r)^2}. \quad (31)$$

Note that in the last integral on the r.h.s. the denominator $(p_1+r)^2$ is missing. As $\delta \to 0$, this integral gives $C^{(2)} \; [(p_1+r)^2]$ which cancels the corresponding term in (28).

Careful analysis of the contributions (30), (31) shows that what we need, in addition to the formulae presented in section 2, is the expansion of the integral $J(4;1,1,2+\delta)$ up to (and including) δ^2 terms. To get it, it is convenient to use the representation (12) with δ substituted by $1+\delta$. The result of this calculation can be presented as

$$J(4;1,1,2+\delta) = \frac{\mathrm{i}\pi^2}{(p_3^2)^{2+\delta}} \frac{1}{xy} \; \frac{1}{1+\delta} \left\{ -\frac{1}{\delta} + (\ln x + \ln y) - \frac{\delta}{2} \left(\ln^2 x + \ln x \ln y + \ln^2 y \right) \right.$$

$$+\frac{\delta^2}{6}\left(\ln^3 x+\ln^2 x\ln y+\ln x\ln^2 y+\ln^3 y\right)$$
$$-\frac{\delta}{2}\left(1-\frac{\delta}{2}(\ln x+\ln y)\right)(1-x-y)\,\Phi^{(1)}(x,y)+\frac{\delta^2}{6}\,\Omega^{(2)}(x,y)+\mathcal{O}(\delta^3)\Bigg\}. \quad (32)$$

Here we define a set of functions $\Omega^{(N)}$ via the derivatives of the functions $\Phi^{(N)}$ (the general case of these functions was considered in [10]) as

$$\Omega^{(N)}(x,y)=\lambda\left[x\frac{\partial}{\partial x}\left(\lambda\Phi^{(N)}(x,y)\right)+y\frac{\partial}{\partial y}\left(\lambda\Phi^{(N)}(x,y)\right)\right]. \quad (33)$$

For $N=1$, the corresponding function $\Omega^{(1)}$ is trivial,

$$\Omega^{(1)}(x,y)=\ln x+\ln y-(x-y)\ln\frac{y}{x}. \quad (34)$$

For $N=2$, the function $\Phi^{(2)}$ is defined by eq. (17), and $\Omega^{(2)}$ can be represented as

$$\Omega^{(2)}(x,y)=\frac{1}{2}\ln x\ln^2\frac{y}{x}+3\int\limits_0^1\frac{\mathrm{d}\xi}{\xi}\left(\ln\frac{y}{x}+2\ln\xi\right)\ln\left(\frac{y\xi^2+(1-x-y)\xi+x}{x}\right), \quad (35)$$

or in terms of polylogarithms,

$$\Omega^{(2)}(x,y)=6\left[\mathrm{Li}_3(-\rho x)+\mathrm{Li}_3(-\rho y)\right]+3\ln\frac{y}{x}\left[\mathrm{Li}_2(-\rho x)-\mathrm{Li}_2(-\rho y)\right]$$
$$-\frac{1}{2}\ln^2\frac{y}{x}\left[\ln(1+\rho x)+\ln(1+\rho y)\right]+\frac{1}{2}\left(\pi^2+\ln(\rho x)\ln(\rho y)\right)\left[\ln(\rho x)+\ln(\rho y)\right]. \quad (36)$$

Note that, combining the derivatives of $\Phi^{(2)}$ in a different way, we may obtain $\Phi^{(1)}$ as

$$\ln\frac{y}{x}\,\Phi^{(1)}(x,y)=\frac{2}{\lambda}\left[(1-x+y)x\frac{\partial}{\partial x}\left(\lambda\Phi^{(2)}(x,y)\right)-(1+x-y)y\frac{\partial}{\partial y}\left(\lambda\Phi^{(2)}(x,y)\right)\right]. \quad (37)$$

Finally, using (32), the integral with irreducible numerator can be represented as

$$C^{(2)}\left[q^2\right]=C^{(2)}\left[(q-r)^2\right]+C^{(2)}\left[r^2\right]-\frac{(i\pi^2)^2}{4\,p_3^2}\left[\ln x+\ln y-(x-y)\ln\frac{y}{x}\right]\Phi^{(1)}(x,y)$$
$$-\frac{(i\pi^2)^2}{2\,p_3^2}\left[\Omega^{(2)}\left(\frac{x}{y},\frac{1}{y}\right)+\Omega^{(2)}\left(\frac{y}{x},\frac{1}{x}\right)\right]+\mathcal{O}(\varepsilon). \quad (38)$$

Note that the factor multiplying $\Phi^{(1)}$ is proportional to $\Omega^{(1)}(x,y)$, eq. (34).

5. Let us consider the non-planar diagram with irreducible numerator, $\widetilde{C}^{(2)}\left[(q+r)^2\right]$. It is easy to see that it is convergent and can be written as

$$\widetilde{C}^{(2)}\left[(q+r)^2\right]=p_2^2\widetilde{C}^{(2)}[1]+\widetilde{C}^{(2)}\left[(p_2-q-r)^2\right]-2\widetilde{C}^{(2)}\left[(p_2,p_2-q-r)\right]. \quad (39)$$

Now, using the symmetry of the non-planar diagram it is possible to see that[7]

$$\widetilde{C}^{(2)}\left[(p_2,p_2-q-r)\right]=\widetilde{C}^{(2)}\left[(p_2,p_1+q+r)\right]. \quad (40)$$

[7]This can be shown using the fact that the non-planar integral (15) remains invariant when we simultaneously change $r\to p_2-r$ and $q\to -p_1-q$.

Writing the l.h.s. of (40) as half the sum of the l.h.s. and the r.h.s., we get

$$\widetilde{C}^{(2)}\left[(p_2, p_2 - q - r)\right] = \frac{1}{2}\,(p_2, p_1 + p_2)\,\widetilde{C}^{(2)}\left[1\right], \tag{41}$$

and the result for the considered integral reduces to

$$\widetilde{C}^{(2)}\left[(q+r)^2\right] = \widetilde{C}^{(2)}\left[(p_2 - q - r)^2\right] - (p_1 p_2)\widetilde{C}^{(2)}\left[1\right]. \tag{42}$$

According to (21) (see also [9]), $\widetilde{C}^{(2)}\left[(p_2 - q - r)^2\right]$ can be represented in terms of the planar diagram as $p_3^2\, C^{(2)}\left[1\right]$. So, finally we arrive at the following result:

$$\widetilde{C}^{(2)}\left[(q+r)^2\right] = p_3^2\, C^{(2)}(p_1^2, p_2^2, p_3^2) - (p_1 p_2)\, \widetilde{C}^{(2)}(p_1^2, p_2^2, p_3^2) \tag{43}$$

where the integrals contributing to the r.h.s. are defined by eqs. (14) and (15). This equation is illustrated by Fig. 3 where "-1" means that the irreducible numerator can be considered as a negative power of the corresponding denominator. It is interesting that we have a "mixture" of different topologies on the r.h.s. of eq. (43) and Fig. 3.

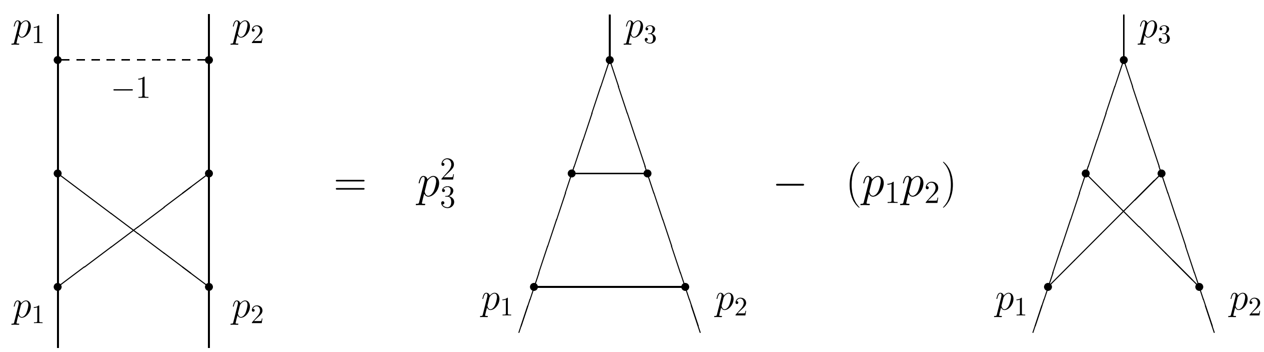

Figure 3: Result for the non-planar diagram with irreducible numerator

6. In this section we shall demonstrate that the problem of tensor decomposition of two-loop three-point functions also requires integrals with irreducible numerators.

Let us consider the case with one free Lorentz index. It is easy to show [22, 5] that

$$C^{(2)}\left[q_\mu\right] = \Delta^{-1}\left\{\left(p_2^2 p_{1\mu} - (p_1p_2)p_{2\mu}\right) C^{(2)}\left[(p_1 q)\right] + \left(p_1^2 p_{2\mu} - (p_1p_2)p_{1\mu}\right) C^{(2)}\left[(p_2 q)\right]\right\} \tag{44}$$

where Δ is nothing but the Källen function, $\Delta = p_1^2 p_2^2 - (p_1p_2)^2 = (p_3^2)^2\lambda^2(x,y)$. The same as (44) is valid for $\widetilde{C}^{(2)}\left[q_\mu\right]$ (we only need to change $C^{(2)} \to \widetilde{C}^{(2)}$ on the r.h.s.). Moreover, the results for $C^{(2)}\left[r_\mu\right]$ and $\widetilde{C}^{(2)}\left[r_\mu\right]$ can be obtained by changing q into r everywhere.

For the two-loop planar diagram, we express $(p_1 q)$ and $(p_2 q)$ in terms of the denominators of the diagram in Fig. 2a, and we get irreducible numerators (27),

$$C^{(2)}\left[(p_1 q)\right] = \frac{1}{2}\left\{-p_1^2 C^{(2)}\left[1\right] + C^{(2)}\left[(p_1+q)^2\right] - C^{(2)}\left[q^2\right]\right\}, \tag{45}$$

$$C^{(2)}\left[(p_2 q)\right] = \frac{1}{2}\left\{p_2^2 C^{(2)}\left[1\right] - C^{(2)}\left[(p_2-q)^2\right] + C^{(2)}\left[q^2\right]\right\}. \qquad (46)$$

Note that in $C^{(2)}\left[(p_3 q)\right]$ the irreducible integrals $C^{(2)}\left[q^2\right]$ disappear. In the case of $C^{(2)}\left[r_\mu\right]$, we do not have irreducible numerators either, because q^2 in (45)–(46) should be replaced by r^2, and the corresponding integral (26) is trivial.

For the non-planar case, we need to express $(p_1 q)$ and $(p_2 q)$ in terms of denominators of the crossed diagram shown in Fig. 2b. So, we get

$$\widetilde{C}^{(2)}\left[(p_1 q)\right] = \frac{1}{2}\left\{-p_1^2 \widetilde{C}^{(2)}\left[1\right] + \widetilde{C}^{(2)}\left[(p_1+q)^2\right] - \widetilde{C}^{(2)}\left[q^2\right]\right\}, \qquad (47)$$

$$\widetilde{C}^{(2)}\left[(p_2 q)\right] = \frac{1}{2}\left\{-\widetilde{C}^{(2)}\left[(p_2-q-r)^2\right] + \widetilde{C}^{(2)}\left[(p_2-r)^2\right] - \widetilde{C}^{(2)}\left[r^2\right] + \widetilde{C}^{(2)}\left[(q+r)^2\right]\right\}, \quad (48)$$

and we see that the irreducible numerator $(q+r)^2$ appears in (48). Analogous results for $\widetilde{C}^{(2)}\left[(p_1 r)\right]$ and $\widetilde{C}^{(2)}\left[(p_2 r)\right]$ can be obtained from (47)–(48) by using the symmetry of the non-planar diagram.

When we consider more free Lorentz indices, $C^{(2)}\left[q_\mu q_\sigma\right], C^{(2)}\left[q_\mu r_\sigma\right]$, etc., we need integrals with higher powers of irreducible numerators. Such integrals will be considered in more detail in [23].

7. In the present paper we have considered an approach to the calculation of three-point two-loop diagrams (Fig. 1a,b) with irreducible numerators (27) which correspond to the dashed lines of auxiliary four-point diagrams shown in Fig. 2a,b. We have obtained exact results for both planar (Fig. 1a) and non-planar (Fig. 1b) cases. The result for the non-planar case can be expressed in terms of the functions corresponding to the diagrams without numerators, as shown in Fig. 3. At the same time, the planar case is more complicated, and the result (38) contains a new function $\Omega^{(2)}$ which can be represented in terms of the derivatives of the function $\Phi^{(2)}$ (see (33)). So, the full set of "non-trivial" functions occurring in two-loop three-point calculations (with massless internal particles and off-shell external momenta) are:

$\Phi^{(1)}$, eqs. (6)–(7), corresponding to the one-loop triangle in four dimensions (4);

$\Psi^{(1)}$, eq. (11) (and eq. (29) of [9]), ε-part of the one-loop triangle;

$\Phi^{(2)}$, eqs. (17) (and eq. (19) of [10]), corresponding to the two-loop planar diagram;

$\Omega^{(2)}$, eqs. (35)–(36), arising in the two-loop planar diagram with irreducible numerator.

It is interesting that all these functions can be obtained from expansions of dimensionally or analytically regularized one-loop integrals in the regulating parameters (ε or δ), see eqs. (10), (13), (32). All of them can be expressed in terms of polylogarithms. Note that the results for some other diagrams can also be expressed in terms of these functions, for example: one- and two-loop off-shell four-point functions with massless internal lines [8], one-loop diagrams with higher number of external lines [24], two-loop massive vacuum diagrams [3, 25].

We have also discussed the problem of tensor reduction of two-loop three-point integrals which also requires integrals with irreducible numerators. A recursive procedure for calculating integrals with higher powers of irreducible numerators can be constructed [23]. Such integrals are needed for the decomposition of tensor integrals with a larger number of uncontracted Lorentz indices.

Important applications of the obtained results are connected with the large momentum expansion of two-loop three-point massive diagrams, and with the calculation of vertex corrections in massless QCD.

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